Feedback Control

Will Wheeler

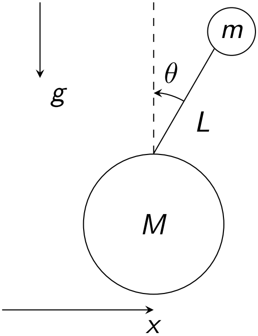
Algorithms Interest Group

Nov 11, 2016

How do Segways stay up?

Segway

Model



Outline

- Dynamical systems and stability
- Controller design (state feedback)
- Observer design (state estimator)
- Integral action
- Simulation

Dynamical Systems

- Model equations

$$\dot{\mathbf{x}} = f(\mathbf{x}, u) \tag{1}$$

- Equilibrium

$$f(\mathbf{x}, u) = 0 \tag{2}$$

- Linearize

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u \tag{3}$$

$$y = \mathbf{C}\mathbf{x} \tag{4}$$

Dynamical Systems - Cart and Pendulum

- Model equations

$$\ddot{x} = \frac{-mg\sin\theta + mL\dot{\theta}^2\sin\theta + F}{M + m\sin^2\theta} \tag{5}$$

$$\ddot{\theta} = \left(\frac{(M+m)g}{L} - m\dot{\theta}^2\cos\theta - F\frac{\cos\theta}{L\sin\theta}\right)\frac{\sin\theta}{M + m\sin^2\theta} \tag{6}$$

- Equilibrium

$$\dot{x} = \ddot{x} = \dot{\theta} = \ddot{\theta} = 0 \Rightarrow \theta = 0 \tag{7}$$

- Linearize

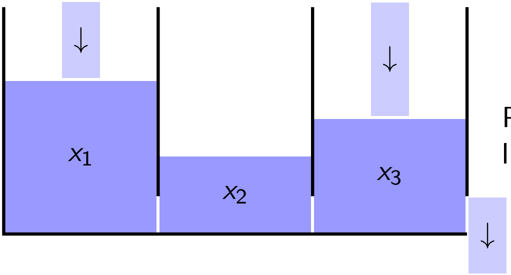
$$\begin{pmatrix}\dot{x}\\ \ddot{x}\\ \dot{\theta}\\ \ddot{\theta}\end{pmatrix} = \begin{pmatrix}0 & 1 & 0 & 0\\ 0 & 0 & -\frac{mg}{M} & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -\frac{(M+m)g}{ML} & 0\end{pmatrix}\begin{pmatrix}x\\ \dot{x}\\ \theta\\ \dot{\theta}\end{pmatrix} + \begin{pmatrix}0\\ \frac{1}{M}\\ 0\\ -\frac{1}{ML}\end{pmatrix}F \tag{8}$$

Dynamical Systems - Three Tanks

Multiple input, multiple output system with hidden state.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_3 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$



Pressure is $\rho g x$, good linear approximation.

Stability

When is an equilibrium $\mathbf{x}_0$ stable?

- stable: perturbations dynamically return to $\mathbf{x}_0$
- unstable: perturbations dynamically grow away from $\mathbf{x}_0$

When is a dynamical system stable?

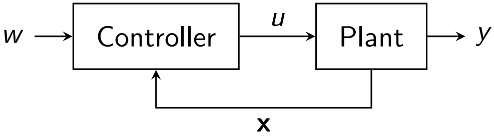
- Diagonalize the system $\rightarrow$ decoupled dynamics

$$\dot{z}_i = \lambda_i z_i \rightarrow z_i = e^{\lambda_i t} \tag{9}$$

- System is stable if all eigenvalues are in the **left half plane**

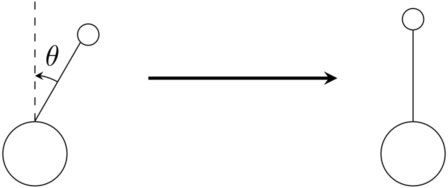
$$\Re(\lambda_i) < 0 \tag{10}$$

Feedback Control

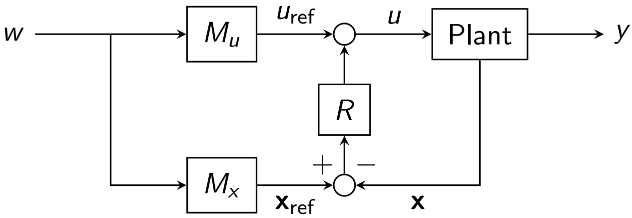


Goals:

- Stabilize an operating point $\mathbf{x}^*$
- Reach $\mathbf{x}^*$ as quickly as possible
- Reach $\mathbf{x}^*$ with few and small oscillations



Feedback Control



Feedforward: $\mathbf{F} = \mathbf{M}_u + \mathbf{R}\mathbf{M}_x$, Feedback: $\mathbf{R}$

Open loop

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}^\top \mathbf{x}$$

Closed loop

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{R})\mathbf{x} + \mathbf{B}\mathbf{F}w$$
$$y = \mathbf{C}^\top \mathbf{x}$$

Controllability

Controllability matrix:

$$\mathbf{Q}_c = \begin{pmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \dots & \mathbf{A}^{n-1}\mathbf{B} \end{pmatrix} \tag{11}$$

System is **controllable** if $\mathbf{Q}_c$ has full rank n

Controller canonical form:

$$\mathbf{A}_c = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{pmatrix} \quad \mathbf{B}_c = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \tag{12}$$

Controller Canonical Form (CCF)

How can we find the transformation $\mathbf{T}$?

Transformation: $\mathbf{A}_c = \mathbf{TAT}^{-1}, \quad \mathbf{B}_c = \mathbf{TB}, \quad \mathbf{C}_c^\top = \mathbf{C}^\top\mathbf{T}$

$$\mathbf{t}_k^\top = \mathbf{t}_{k-1}^\top\mathbf{A} \qquad \mathbf{t}_k^\top\mathbf{B} = 0, \quad \mathbf{t}_n^\top\mathbf{B} = 1$$

$$\mathbf{t}_k^\top = \mathbf{t}_1^\top\mathbf{A}^{k-1} \qquad \mathbf{t}_1^\top\mathbf{A}^{k-1}\mathbf{B} = 0, \quad \mathbf{t}_1^\top\mathbf{A}^{n-1}\mathbf{B} = 1$$

$$\mathbf{t}_1^\top[\mathbf{B}, \mathbf{AB}, \mathbf{A}^2\mathbf{B}, \dots, \mathbf{A}^{n-1}\mathbf{B}] = \mathbf{t}_1^\top\mathbf{Q}_c = \mathbf{e}_n^\top \tag{13}$$

$$\mathbf{TA}_c = \begin{pmatrix} \mathbf{t}_1^\top \\ \mathbf{t}_2^\top \\ \vdots \\ \mathbf{t}_n^\top \end{pmatrix} \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{pmatrix} = \begin{pmatrix} \mathbf{t}_2^\top \\ \mathbf{t}_3^\top \\ \vdots \\ \times \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{t}_1 & \mathbf{t}_2 & \dots & \mathbf{t}_n \end{pmatrix}^\top \mathbf{B} = \begin{pmatrix} 0 & 0 & \dots & 1 \end{pmatrix}^\top$$

Controller Design

- Pick **desired** closed-loop eigenvalues (left half plane!)

$$\begin{aligned} p(x) &= (x-\lambda_1^*)(x-\lambda_2^*)\cdots(x-\lambda_n^*) \\ &= x^n + p_{n-1}x^{n-1} + \ldots + p_1 x + p_0 \end{aligned}$$

- Transform to CCF. Characteristic polynomial is

$$p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \tag{14}$$

- Feedback matrix $\mathbf{R}_c$ (in CCF)

$$r_i = p_i - a_i \tag{15}$$

- Feedforward $\mathbf{M}_x$ and $\mathbf{M}_u$: in steady-state, $\dot{\mathbf{x}} = 0$ and $y = w$

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C}^\top & 0 \end{pmatrix} \begin{pmatrix} \mathbf{M}_x \\ \mathbf{M}_u \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{I} \end{pmatrix}$$

Example: Controller for Cart and Pendulum

Recall we had the form

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & a_{23} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & a_{43} & 0 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 0 \\ b_2 \\ 0 \\ b_4 \end{pmatrix} \tag{16}$$

Set eigenvalues of $\mathbf{A} - \mathbf{BR}$ by matching coefficients p_i.

Answer: $\mathbf{R} = \begin{pmatrix} r_0 & r_1 & r_2 & r_3 \end{pmatrix}$

$$r_0 = -p_0 ML/g$$
$$r_1 = -p_1 ML/g$$
$$r_2 = -(M+m)g - ML\left(p_0 L/g + p_2\right)$$
$$r_3 = -ML\left(p_1 L/g + p_3\right)$$

Controllability matrix $\mathbf{Q}_c$ full rank

$$\begin{pmatrix} 0 & b_2 & 0 & a_{23} b_4 \\ b_2 & 0 & a_{23} b_4 & 0 \\ 0 & b_4 & 0 & a_{43} b_4 \\ b_4 & 0 & a_{43} b_4 & 0 \end{pmatrix}$$

Linear Quadratic Regulator

If there are multiple inputs, equations for $\mathbf{R}$ underdetermined!
Many solutions! How do we choose?
New goal: get to desired state $\mathbf{x}^*$ with minimum energy.
Minimize a cost function $\mathcal{J}(\mathbf{R})$:

$$\mathcal{J}(\mathbf{R}) = \frac{1}{2}\int_0^\infty \mathbf{x}^\top(t)\mathbf{Q}\,\mathbf{x}(t)\,dt + \frac{1}{2}\int_0^\infty \mathbf{u}^\top(t)\widetilde{\mathbf{Q}}\,\mathbf{u}(t)\,dt \qquad (17)$$

$\mathbf{Q} \geq 0$ pos semidef; $\widetilde{\mathbf{Q}} > 0$ pos def.
These weight the physical states and inputs by **cost factors that we can choose**.
Answer:

$$\mathbf{R} = -\widetilde{\mathbf{Q}}^{-1}\mathbf{B}^\top\mathbf{P} \qquad (18)$$

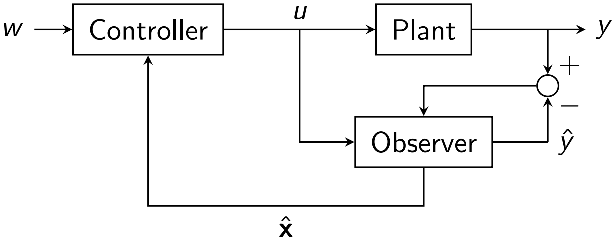
where $\mathbf{P}$ satisfies the Ricatti equation

$$\mathbf{P}\mathbf{B}\widetilde{\mathbf{Q}}^{-1}\mathbf{B}^\top\mathbf{P} - \mathbf{P}\mathbf{A} - \mathbf{A}^\top\mathbf{P} - \mathbf{Q} = 0 \qquad (19)$$

Observer

What if we can't measure all the states?

1. Run a simulation of the plant in parallel
2. Compare outputs
3. Feed the difference back to the simulation



$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}(y - \hat{y})$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$\begin{aligned} \dot{\hat{\mathbf{x}}} - \dot{\mathbf{x}} &= \mathbf{A}(\hat{\mathbf{x}} - \mathbf{x}) + \mathbf{L}(y - \hat{y}) \\ &= \mathbf{A}(\hat{\mathbf{x}} - \mathbf{x}) - \mathbf{L}\mathbf{C}^\top(\hat{\mathbf{x}} - \mathbf{x}) \end{aligned}$$

Controller-Observer Duality

$$\begin{pmatrix} \dot{\mathbf{x}} \\ y \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C}^\top & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ u \end{pmatrix} \tag{20}$$

- Closed loop: $\mathbf{A} - \mathbf{BR}$
- Set eigenvalues by choosing $\mathbf{R}$
- Controllability matrix $\mathbf{Q}_c$ $\begin{pmatrix} \mathbf{B} & \mathbf{AB} & \dots & \mathbf{A}^{n-1}\mathbf{B} \end{pmatrix}$

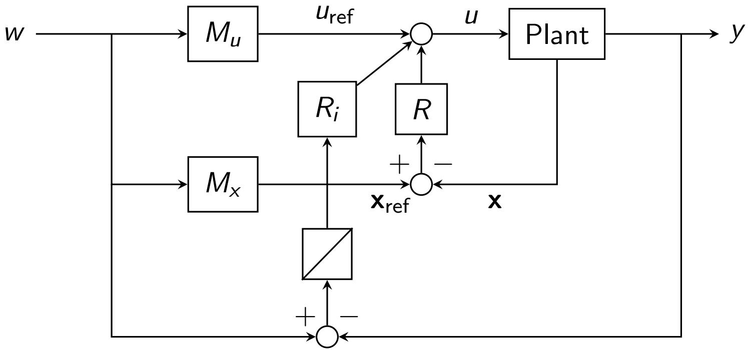
- Closed loop: $\mathbf{A}^\top - \mathbf{CL}^\top$
- Set eigenvalues by choosing $\mathbf{L}$
- Observability matrix $\mathbf{Q}_o$ $\begin{pmatrix} \mathbf{C}^\top & \mathbf{C}^\top\mathbf{A}^\top & \dots & \mathbf{C}^\top(\mathbf{A}^\top)^{n-1} \end{pmatrix}$

New dynamics $\rightarrow$ more eigenvalues.

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}} \end{pmatrix} = \begin{pmatrix} \mathbf{A} & -\mathbf{BR} \\ \mathbf{LC}^\top & \mathbf{A} - \mathbf{LC}^\top - \mathbf{BR} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{pmatrix} \tag{21}$$

Integral Action

Compensate constant disturbances and model errors.



$$\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}_i \end{pmatrix} = \begin{pmatrix} \mathbf{A} - \mathbf{B}\mathbf{R} & \mathbf{B}\mathbf{R}_i \\ -\mathbf{C}^\top & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{x}_i \end{pmatrix} + \begin{pmatrix} \mathbf{B}\mathbf{F} \\ \mathbf{I} \end{pmatrix} w \qquad (22)$$