Algorithms Discussion group

An Introduction To Quantum Algorithms

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Outline

- What are quantum algorithms?
- Examples
- Some background building (quantum registers and logic gates)
- Grover's Algorithm
- Example of Grover's algorithm
- Conclusion

What are quantum algorithms?

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Simulating Physics with Computers

Richard P. Feynman

Department of Physics, California Institute of Technology, Pasadena, California 91107

Received May 7, 1981

1. INTRODUCTION

On the program it says this is a keynote speech—and I don't know what a keynote speech is. I do not intend in any way to suggest what should be in this meeting as a keynote of the subjects or anything like that. I have my own things to say and to talk about and there's no implication that anybody needs to talk about the same thing or anything like it. So what I want to talk about is what Mike Dertouzos suggested that nobody would talk about. I want to talk about the problem of simulating physics with computers and I mean that in a specific way which I am going to explain. The reason for doing this is something that I learned about from Ed Fredkin, and my entire interest in the subject has been inspired by him. It has to do with learning something about the possibilities of computers, and

- Richard Feynman pointed out that accurately and efficiently simulating quantum mechanical systems would be impossible on a classical computer, but that a new kind of machine, a computer itself built of quantum mechanical elements which obey quantum mechanical laws, might one day perform efficient simulations of quantum systems.

- Classical computers are inherently unable to simulate such a system using sub-exponential time and space complexity due to the exponential growth of the amount of data required to completely represent a quantum system.

- To simulate quantum systems we need efficient quantum algorithms.

- Quantum algorithms can also give huge speed up in solving classical problems of search etc.

Examples

- There have been many algorithms that have been developed for quantum computers.

<u>Few examples</u>

- Deutsch–Jozsa algorithm
- Simon's algorithm
- Quantum phase estimation algorithm
- Grover’s Algorithm
- Shorts algorithm
- Hidden subgroup problem
- Boson sampling problem

<u>Today We’ll talk about</u>

- Grover’s Algortihm

Background Building

- A quantum bit or “qubit” is just a complex superposition of classical bits.

$$|\psi\rangle = a_0 |0\rangle + a_1 |1\rangle|$$

- We will be talking in terms of computational basis i.e

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Quantum registers are bit strings whose length determines the amount of information they can store. The state space of a size-n quantum register is :

$$|\psi_n\rangle = \sum_{i=0}^{2^n-1} a_i |i\rangle$$

- Eg for n =3, a quatum register is

$$\||\psi_2\rangle = a_0 |000\rangle + a_1 |001\rangle + a_2 |010\rangle + a_3 |011\rangle + a_4 |100\rangle + a_5 |101\rangle + a_6 |110\rangle + a_7 |111\rangle$$

Background Building

- Quantum Logic Gates : These are basically operators that evolve the states.
- One important example is the single-qubit Hadamard operator (Fair coin flip operator)

$$
-\boxed{H}- = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \langle 0| + \frac{|0\rangle - |1\rangle}{\sqrt{2}} \langle 1|
$$

$$
H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \langle 0|0\rangle + \frac{|0\rangle - |1\rangle}{\sqrt{2}} \langle 1|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}
$$

$$
H|1\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \langle 0|1\rangle + \frac{|0\rangle - |1\rangle}{\sqrt{2}} \langle 1|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}
$$

- The probability distributions of the two results are exactly the same, the two states differing only by the phase of |1>.
- Notice what H does to a |0> state. We'll use this later.

Grover's Algorithm

- It was developed by an Indian mathematician Lov Grover in 1996.
- Grover's algorithm deals with searching a unique element in an unordered list.
- Classically this takes O(N) time.
- On a quantum computer, Grover's algorithm does this search in $O(\sqrt{N})$ operations.

THE ALGORITHM

Grover's ONE Iteration step

1. Initializing the state to all zeros.
2. Converting it to a uniform superposition
3. Applying the quantum ORACLE operator
4. Amplitude amplification

One Iteration

1. Begin with n qubits all initialized to 0.

$$|0\rangle^{\otimes n} = |0\rangle$$

2. Apply the n dimensional Hadamard operator which converts this into a uniform superposition of all possible 2^n states, with amplitude of each being $1/\sqrt{2^n}$

$$|\psi\rangle = H^{\otimes n} |0\rangle^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle$$

Recall that we're looking for a unique element. We don't necessarily know that element but can map it to a function such that f(x) = 1 for the unique element and f(x) = 0 otherwise.

1. Now the apply the Oracle operator.
2. Oracle is basically a black-box function, and this quantum oracle is a quantum black-box, meaning it can observe and modify the system without collapsing it to a classical state.

$$|x\rangle \xrightarrow{\mathcal{O}} (-1)^{f(x)} |x\rangle$$

One Iteration cont.

Amplitude amplification involved 2 steps.

1. Conditional phase shift of all elements except |0> . This is done by applying $2|0\rangle\langle 0| - I$
2. Another application of Hadamard gate. $H^{\otimes n}[2|0\rangle\langle 0| - I]H^{\otimes n}$

This would give $2|\psi\rangle\langle\psi| - I$

One Iteration cont.

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To Summarize

1. $\|0\rangle^{\otimes n}$	initial state
2. $H^{\otimes n}\|0\rangle^{\otimes n} = \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}\|x\rangle = \|\psi\rangle$	apply the Hadamard transform to all qubits
3. $[(2\|\psi\rangle\langle\psi\| - I)\mathcal{O}]^R\|\psi\rangle \approx \|x_0\rangle$	apply the Grover iteration $R \approx \frac{\pi}{4}\sqrt{2^n}$ times
4. x_0	measure the register

Small Example

Consider a system of 3 qubits and we're searching for the string 011. i.e. $x_0 = |011>$

Grovers algorithm begins by applying Hadamard operator to |000>

$$H^3 |000\rangle = \frac{1}{2\sqrt{2}} |000\rangle + \frac{1}{2\sqrt{2}} |001\rangle + \ldots + \frac{1}{2\sqrt{2}} |111\rangle = \frac{1}{2\sqrt{2}} \sum_{x=0}^{7} |x\rangle = |\psi\rangle$$

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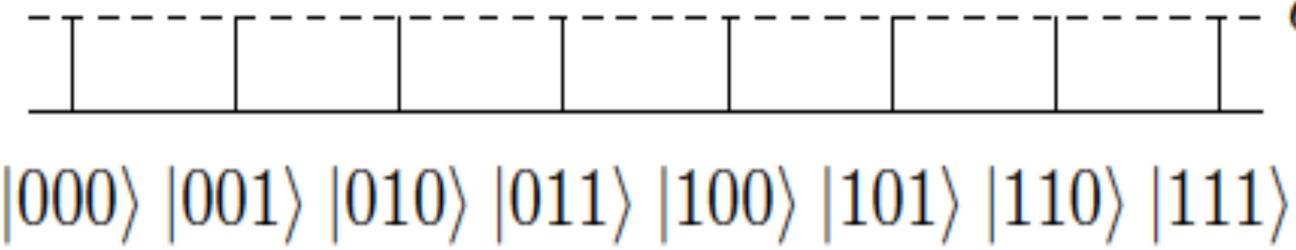
Geometrically the equal superposition of states resulting from the first Hadamard transform look like



$$\alpha_\psi = \frac{1}{2\sqrt{2}}$$

When Oracle is called

$$|x\rangle = \frac{1}{2\sqrt{2}}|000\rangle + \frac{1}{2\sqrt{2}}|001\rangle + \frac{1}{2\sqrt{2}}|010\rangle - \frac{1}{2\sqrt{2}}|011\rangle + \ldots + \frac{1}{2\sqrt{2}}|111\rangle$$

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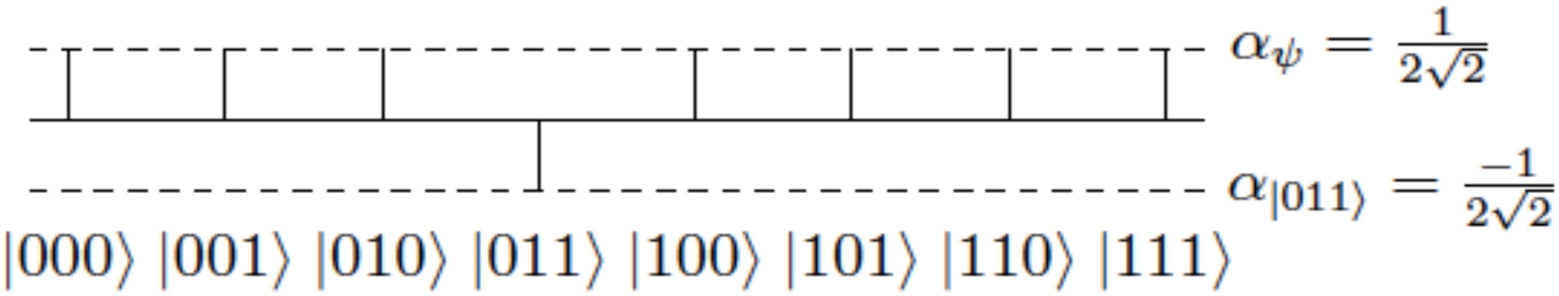

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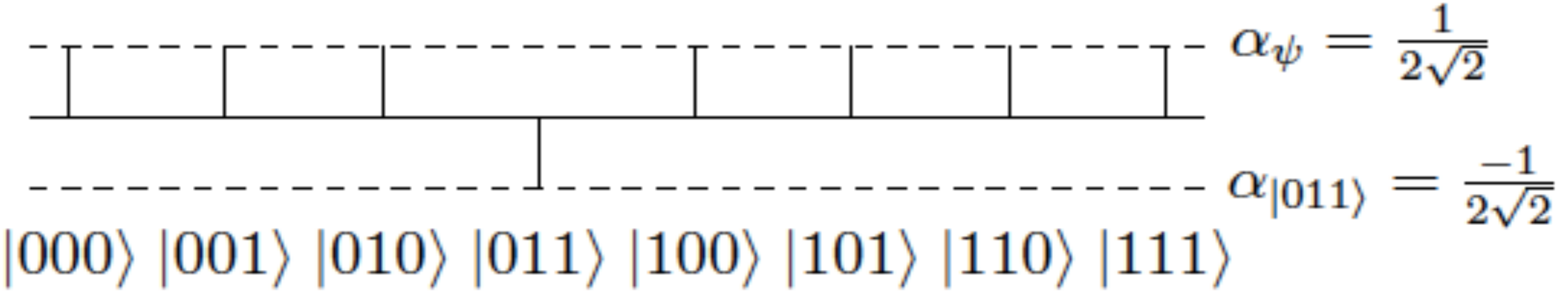

When amplitude amplification is applied

$$[2|\psi\rangle\langle\psi| - I]|x\rangle$$

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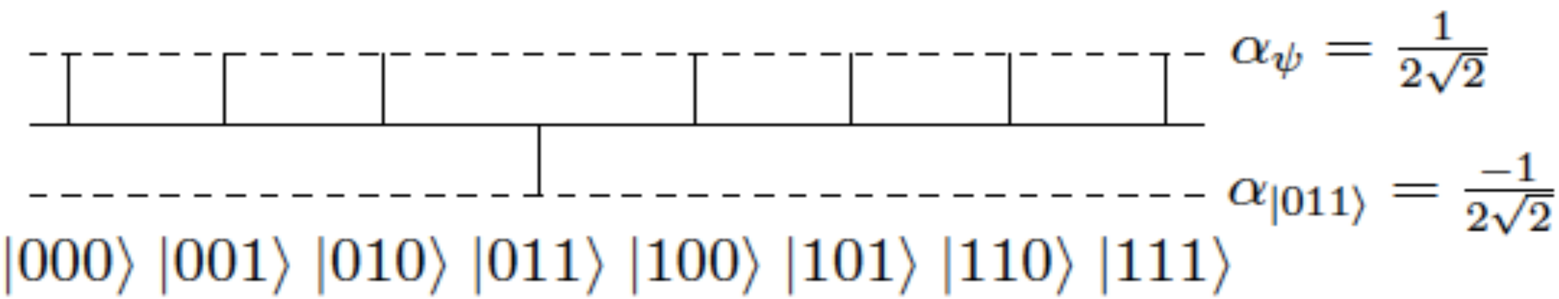


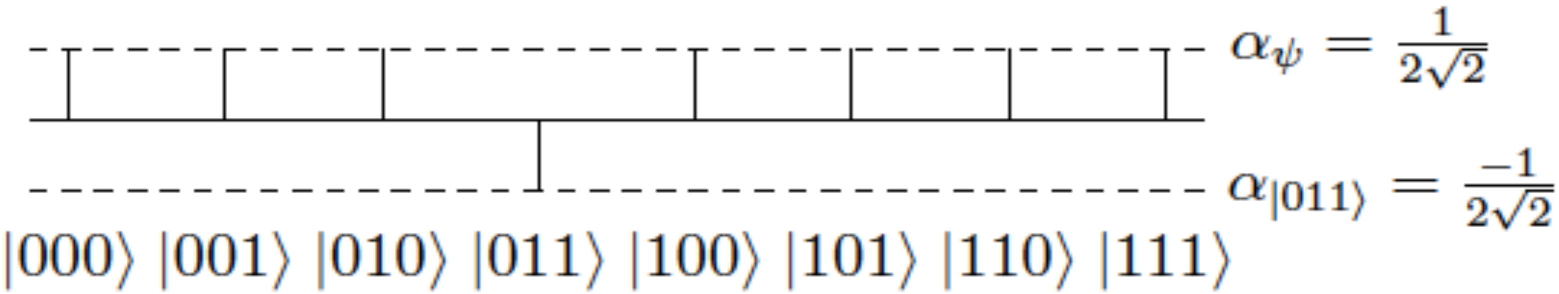
When amplitude amplification is applied

$$[2|\psi\rangle\langle\psi| - I]\,|x\rangle$$

$$\models [2|\psi\rangle\langle\psi| - I]\left[|\psi\rangle - \frac{2}{2\sqrt{2}}|011\rangle\right]$$

When Oracle is called

$$|x\rangle = \frac{1}{2\sqrt{2}}|000\rangle + \frac{1}{2\sqrt{2}}|001\rangle + \frac{1}{2\sqrt{2}}|010\rangle - \frac{1}{2\sqrt{2}}|011\rangle + \ldots + \frac{1}{2\sqrt{2}}|111\rangle$$



When amplitude amplification is applied

$$[2|\psi\rangle\langle\psi| - I]|x\rangle$$

$$= [2|\psi\rangle\langle\psi| - I]\left[|\psi\rangle - \frac{2}{2\sqrt{2}}|011\rangle\right]$$

$$= 2|\psi\rangle\langle\psi|\psi\rangle - |\psi\rangle - \frac{2}{\sqrt{2}}|\psi\rangle\langle\psi|011\rangle + \frac{1}{\sqrt{2}}|011\rangle$$

$$= |\psi\rangle - \frac{1}{2}|\psi\rangle + \frac{1}{\sqrt{2}}|011\rangle$$

In terms of the equal superposition state

$$=\frac{1}{2}\left[\frac{1}{2\sqrt{2}}\sum_{x=0}^{7}|x\rangle\right]+\frac{1}{\sqrt{2}}|011\rangle$$

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$$=\frac{1}{4\sqrt{2}}\sum_{\substack{x=0\\x\neq 3}}^{7}|x\rangle+\frac{1}{4\sqrt{2}}|011\rangle+\frac{1}{\sqrt{2}}|011\rangle$$

In terms of the equal superposition state

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$$= \frac{1}{4\sqrt{2}}\sum_{\substack{x=0\\x\neq 3}}^{7}|x\rangle + \frac{5}{4\sqrt{2}}|011\rangle$$

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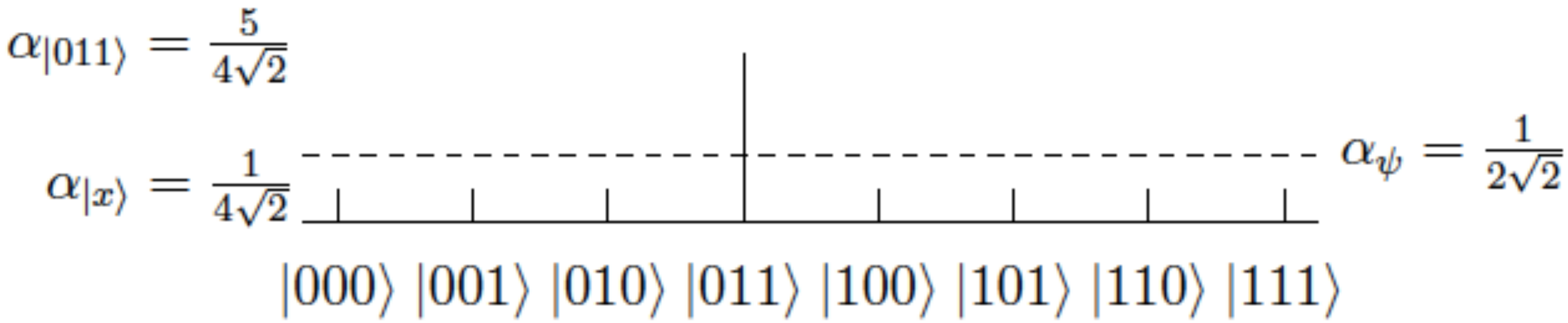
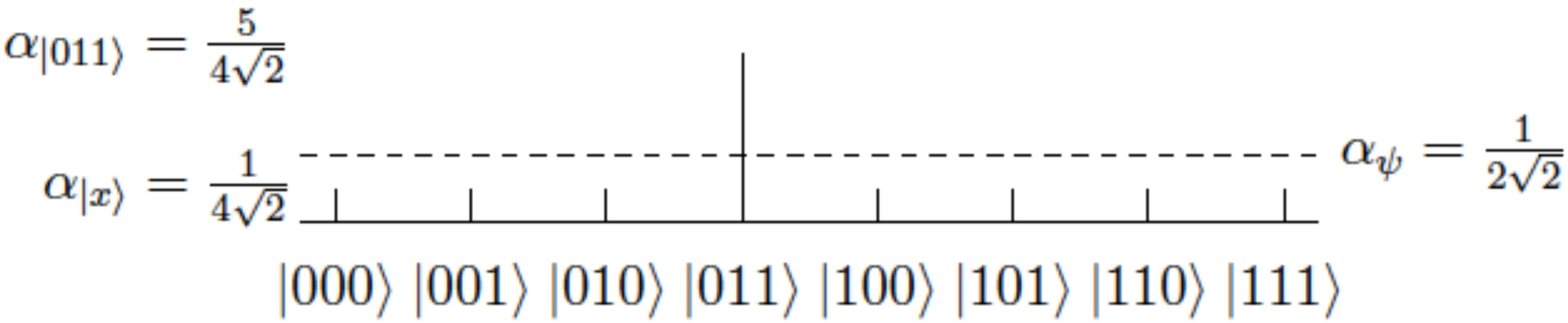


First iteration done!



First iteration done!

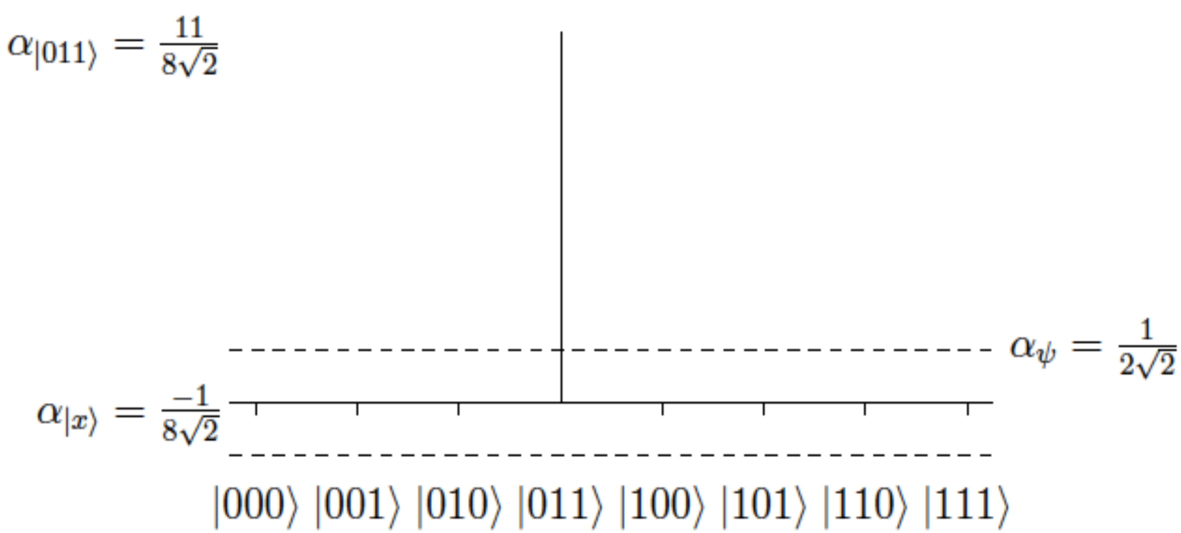
Now if we apply another iteration

$$
\begin{aligned}
|x\rangle &= \frac{1}{4\sqrt{2}}|000\rangle + \frac{1}{4\sqrt{2}}|001\rangle + \frac{1}{4\sqrt{2}}|010\rangle - \frac{5}{4\sqrt{2}}|011\rangle + \ldots + \frac{1}{4\sqrt{2}}|111\rangle \\
&= \frac{1}{4\sqrt{2}}\sum_{\substack{x=0 \\ x\neq 3}}^{7}|x\rangle - \frac{5}{4\sqrt{2}}|011\rangle \\
&= \frac{1}{4\sqrt{2}}\sum_{x=0}^{7}|x\rangle - \frac{6}{4\sqrt{2}}|011\rangle \\
&= \frac{1}{2}|\psi\rangle - \frac{3}{2\sqrt{2}}|011\rangle
\end{aligned}
$$

After some simple math

$$
\begin{aligned}
&[2|\psi\rangle\langle\psi| - I]\left[\frac{1}{2}|\psi\rangle - \frac{3}{2\sqrt{2}}|011\rangle\right] \\
&= 2\left(\frac{1}{2}\right)|\psi\rangle\langle\psi|\psi\rangle - \frac{1}{2}|\psi\rangle - 2\left(\frac{3}{2\sqrt{2}}\right)|\psi\rangle\langle\psi|011\rangle + \frac{3}{2\sqrt{2}}|011\rangle \\
&= |\psi\rangle - \frac{1}{2}|\psi\rangle - \frac{3}{\sqrt{2}}\left(\frac{1}{2\sqrt{2}}\right)|\psi\rangle + \frac{3}{2\sqrt{2}}|011\rangle \\
&= -\frac{1}{4}|\psi\rangle + \frac{3}{2\sqrt{2}}|011\rangle \\
&= -\frac{1}{4}\left[\frac{1}{2\sqrt{2}}\sum_{\substack{x=0\\x\neq 3}}^{7}|x\rangle + \frac{1}{2\sqrt{2}}|011\rangle\right] + \frac{3}{2\sqrt{2}}|011\rangle \\
&= -\frac{1}{8\sqrt{2}}\sum_{\substack{x=0\\x\neq 3}}^{7}|x\rangle + \frac{11}{8\sqrt{2}}|011\rangle
\end{aligned}
$$

$$|x\rangle = -\frac{1}{8\sqrt{2}}|000\rangle - \frac{1}{8\sqrt{2}}|001\rangle - \frac{1}{8\sqrt{2}}|010\rangle + \frac{11}{8\sqrt{2}}|011\rangle - \ldots - \frac{1}{8\sqrt{2}}|111\rangle$$



Now when we measure

Now when the system is observed, the probability that the state representative of the correct solution, |011>, will be measured is 94:5%. The probability of finding an incorrect state is 5.5%; Grover's algorithm is more than 17 times more likely to give the correct answer than an incorrect one with an input size of N = 8, and the error only decreases as the input size increases.

Although Grover’s algorithm is probabilistic, the error truly becomes negligible as N grows large.

Thanks!

- Referrences :
https://people.cs.umass.edu/~strubell/doc/quantum_tutorial.pdf
- https://www.cs.cmu.edu/~odonnell/quantum15/lecture04.pdf
- https://en.wikipedia.org/wiki/Grover%27s_algorithm